UNIVERSITY OF TEXAS AT AUSTIN

Stat/Discrete Methods for Sci Computing Problem Set 1

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1 EXERCISE 1

(1) Given the distribution $\frac{1}{3\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{x}{3})2}$ what is the probability that $x > 1$?
Solution:

$$\begin{aligned}
P(x>1) &= \int_1^\infty \frac{1}{3\sqrt{2\pi}} x e^{-\frac{1}{2}(\frac{x}{3})^2} dx \\
&= \frac{1}{3\sqrt{2\pi}} \int_1^\infty \frac{1}{2} e^{-\frac{1}{18}x^2} dx^2 \qquad (y \doteq \frac{1}{18}x^2) \\
&= \frac{3}{\sqrt{2\pi}} \int_{\frac{1}{18}}^\infty -e^{-y} dy \\
&= \frac{3}{\sqrt{2\pi}} e^{-\frac{1}{18}}
\end{aligned}$$

(2) Compare the Markov and Chebyshev bounds for the following probability distributions:

$$\text{a)}\ \ p(x) = \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\text{b)}\ \ p(x) = \begin{cases} \frac{1}{2}, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Solution:
For a), $E(x) = 1$, $Var(x) = 0$. So Markov inequality for a) is $Prob(x \ge a) \le \frac{1}{a}$, while Chebyshev inequality for a) vanishes to $Prob(|x-1| > 0) = 0$.

For b), $E(x) = \int_0^2 \frac{1}{2} x dx = 1$, $Var(x) = \int_0^2 \frac{1}{2}(x-1)^2 dx = \frac{1}{3}$. So Markov inequality for b) is $Prob(x \geq a) \leq \frac{1}{a}$ (which is the same as in problem a)), while Chebyshev inequality for b) is $Prob(|x-1| > \sqrt{\frac{1}{3}} a) \leq \frac{1}{a^2}$.

(3) Let s be the sum of n independent ransom variables $x_1, x_2, \cdots, x_n$ where for each i,

$$x_i = \begin{cases} 0, & \text{Prob is } p \\ 1, & \text{Prob is } 1-p \end{cases}$$

How large must δ be if we wish to have $\text{Prob}(s > (1+\delta)n) < \epsilon$

Solution:
$E(x_i) = 1-p$, $Var(x) = (1-(1-p))^2(1-p) + (0-(1-p))^2 p = p^2(1-p) + (1-p)^2 p = p^2 - p^3 + p - 2p^2 + p^3 = p - p^2$.
So $Prob(x \geq a) \leq \frac{1-p}{a}$ for each i. So $E(s) = n(1-p)$, $Var(s) = np(1-p)$.
Applying Markov inequality we get $\text{Prob}(x \geq n(1+\delta)) \leq \frac{n(1-p)}{n(1+\delta)} = \epsilon$. So $\delta = \frac{1-p}{\epsilon} - 1$
Applying Chebyshev inequality we get $\text{Prob}(|s - n(1-p)| > anp(1-p)) \leq \frac{1}{a^2} = \epsilon$. So $a = \sqrt{\frac{1}{\epsilon}}$
$\text{Prob}(x < n(1-p) - \sqrt{\frac{1}{\epsilon}} np(1-p) \leq \epsilon$. Therefore take $\delta \doteq 1 - (1-p)(1 - \sqrt{\frac{1}{\epsilon}} p)$. We have for sure that $\text{Prob}(x < n(1-\delta)$ $=\text{Prob}(x < n(1-p)(1-\sqrt{\frac{1}{\epsilon}} p) \leq \epsilon$.

2 EXERCISE 2

(1) For what values of d do area, $A(d)$, and the volume, $V(d)$, of a d-dimensional unit sphere take on their maximum values?

$$\begin{aligned} A(d) &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \\ \frac{A(d+2)}{A(d)} &= \frac{2\pi}{d} \begin{cases} < 1, & d \geq 7 \\ > 1, & d \leq 6 \end{cases} \end{aligned}$$

This means $\cdots < A(11) < A(9) < A(7) > A(5) > A(3)$ and $\cdots < A(10) < A(8) > A(6) > A(4)$. $A(7)$ and $A(8)$ are the largest $A(d)$ for respectively odd d and even d. While $A(7) = \frac{2\pi^{3.5}}{\Gamma(3.5)} \approx 33.07$, and $A(8) = \frac{2\pi^4}{\Gamma(4)} \approx 32.47$. So $A(7) = \frac{2\pi^{3.5}}{\Gamma(3.5)}$ is the largest $A(d)$ for every $d \geq 1$.

$$\begin{aligned} V(d) &= \frac{2\pi^{d/2}}{d\Gamma(\frac{d}{2})} \\ \frac{V(d+2)}{V(d)} &= \frac{2\pi}{d+2} \begin{cases} < 1, & d \geq 5 \\ > 1, & d \leq 4 \end{cases} \end{aligned}$$

This means $V(5)$ and $V(6)$ are the largest volumes $V(d)$ fo respectively odd d and even d. Turns out $V(5) = \frac{2\pi^{2.5}}{5\Gamma(2.5)} \approx 5.26$ ($> V(6) \approx 5.17$) is the largest volume.

(2) How do the area and the volume of a sphere with *radius*=2 behave as the dimension of the space increases? What if the radius was larger than two but independent of d?
radius=2

$$\begin{aligned} A(d) &= \frac{2\pi^{d/2}2^{d-1}}{\Gamma(\frac{d}{2})} \\ \frac{A(d+2)}{A(d)} &= \frac{8\pi}{d}\begin{cases} <1, & d \geq 26 \\ >1, & d \leq 25 \end{cases} \end{aligned}$$

This means $A(26)$ and $A(27)$ are the largest area $A(d)$ fo respectively even d and odd d. Turns out $A(26) = \frac{2^1 4\pi^{13}}{\Gamma(13)} \approx 4.07 \times 10^5$ ($> V(27) \approx 4.04 \times 10^5$) is the largest area.

$$\begin{aligned} V(d) &= \frac{2\pi^{d/2}2^{d}}{d\Gamma(\frac{d}{2})} \\ \frac{V(d+2)}{V(d)} &= \frac{8\pi}{d+2}\begin{cases} <1, & d \geq 24 \\ >1, & d \leq 23 \end{cases} \end{aligned}$$

This means $V(24)$ and $V(25)$ are the largest volumes $V(d)$ fo respectively even d and odd d. Turns out $V(24) = \frac{2^2 2\pi^{12}}{3\Gamma(12)} \approx 3.24 \times 10^4$ ($> V(25) \approx 3.21 \times 10^4$) is the largest volume.

As we can see from the previous analysis, both area and volume increase with respect to d for smaller d and decrease when d is large. And when radius is larger than 2, this boundary between increase and decrease will be larger, meaning $\text{argmax}_d A(d)$ and $\text{argmax}_d V(d)$ will be larger.

(3) What function of d would the radius need to be for a sphere of radius r to have approximately constant volume as the dimension increases?

$$\begin{aligned} V(d,r) &= \frac{2\pi^{d/2}r^{d}}{d\Gamma(\frac{d}{2})} \\ \frac{V(d+2,r)}{V(d,r)} &= \frac{2\pi r^2}{d+2} = 1 \\ \therefore r &= \sqrt{\frac{d+2}{2\pi}} \\ \text{Then } V(d) &= \begin{cases} \sqrt{\frac{6}{\pi}}, & d \text{ odd} \\ 2, & d \text{ even} \end{cases} \end{aligned}$$

3 Exercise 3

(1) For each of $a = 2$, and 3 give a probability distribution for a nonnegative random variable x were $\text{Prob}(x \geq aE(x)) = \frac{1}{a}$.

Solution:

$$\text{a=2:}\ \ p(x) = \begin{cases} 1/2, & x = 2 \\ 1/2, & x = 0 \end{cases} \quad E(x) = 1.\text{Prob}(x \geq aE(x)) = \text{Prob}(x \geq 2) = \frac{1}{2} = \frac{1}{a}$$

$$\text{a=3:}\ \ p(x) = \begin{cases} 1/3, & x = 3 \\ 2/3, & x = 0 \end{cases} \quad E(x) = 1.\text{Prob}(x \geq aE(x)) = \text{Prob}(x \geq 3) = \frac{1}{3} = \frac{1}{a}$$

(2)

$$p(x) = \begin{cases} 1/a, & x = a \\ 1 - 1/a, & x = 0 \end{cases} \quad E(x) = 1.\text{Prob}(x \geq aE(x)) = \text{Prob}(x \geq a) = \frac{1}{a}$$

4 EXERCISE 4

Suppose sphere 1 is centered at origin, while sphere 2 is centered at $ae_1 = (a, 0, 0, \cdots, 0)$. The intersection becomes

$$\begin{cases} (x_1 - a)^2 + x_2^2 + \cdots + x_d^2 \leq 1 \\ x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1 \end{cases}$$

Consider the cap of S_1

$$\begin{cases} x_1 \geq \frac{a}{2} \\ x_1^2 + x_2^2 + \cdots + x_d^2 \leq 1 \end{cases}$$

It's easy to see this cap is also contained in S_2. $((x_1 - a)^2 + x_2^2 + \cdots + x_d^2 \leq (x_1 - a)^2 + 1 - x_1^2 \leq (\frac{a}{2} - 1)^2 + 1 - (\frac{a}{2})^2 = 1)$ Likewise the cap of S_2

$$\begin{cases} x_1 \leq \frac{a}{2} \\ (x_1 - a)^2 + x_2^2 + \cdots + x_d^2 \leq 1 \end{cases}$$

is also contained in S_1. So by considering the volume of these two caps we can decide the volume of their intersection.
Applying the Lemma with the plane $x_1 = \frac{a}{2}$, $(c = \frac{a\sqrt{d-1}}{2})$ we get the fraction of the volume of the cap is below $\frac{2}{c} e^{-\frac{c^2}{2}} = \frac{4}{a\sqrt{d-1}} e^{-\frac{a^2(d-1)}{8}}$