Math 241 Homework 1

Sayeed Gulmahamad

February 10, 2015

Set Theory

Problem 1 : These are questions on abstract set theory. Assume capital letters are arbitrary sets and Ω the universe for all the following questions. Answer as succinctly as possible

(a) [easy] Answer the following as best as possible.

A $\cup$ A= A

A $\cap$ A= A

A $\cap$ ϕ= ϕ

A $\cup$ Ω= Ω

A $\cap$ Ω= A

A $\cup$ A^c= Ω

ϕ^c= ϕ

Ω^c= A

A\A= ϕ, because if you have A= $\{1,2\}$; A-A or $\{1,2\}$ - $\{1,2\}$ will give you $\{\}$ or ϕ

A\Ω= A^c

A\ϕ= A

(b) [easy] Are the following true (T) or false (F) for arbitrary sets A; B; C? The last one is extra credit and requires an explanation for bonus points.

A $\subseteq$ Ω= F

A $\subset$ Ω= T

ϕ $\subseteq$ A and A $\subseteq$ Ω= F

A $\subseteq$ A $\cup$ B= F

A $\subseteq$ A $\cap$ b= F

A $\in$ A= T, since the statement is comparing two sets that are the same, the elements of the first set will indeed be inside of the second set.

(c) [harder] Are the following true (T) or false (F) for the arbitrary set A?

A $\subseteq$ A= T

$A \subset A$= T
$\phi \subseteq A$= F
$\phi \subset A$= T
$\phi \subseteq \phi$= T
$\phi \subset \phi$= T

harder] Are the following true (T) or false (F)? The symbol $\Rightarrow$ denotes logical implication i.e. if the conditions on the l.h.s are met, the statement on the r.h.s is always true. Commas should be interpreted to mean "and."

$A \subseteq B \Rightarrow A \cap B$=A = F
$A \subseteq B$, $B \subseteq C \Rightarrow A \subseteq C$= T
$A \subseteq B$, $B \subseteq C \Rightarrow A \subset c$= F
$A \subseteq B$, $A \subseteq C \Rightarrow A \subset B \cap C$= F
$A \subset A \cup B$= T

(e) [harder] Express $A \cap B$ only in terms of set subtraction (by using the symbol "\").

$A \cap B$= $2^A/2^B$

(f) [easy] Explain why $A \cup B$= $B \cup A$ in English.

$A \cup B$= $B \cup A$ because the "Union" operator ($\cup$) is similar to addition, or combining the sets. Therefore, the order of the sets when using the Union operator does not matter. In addition, one trick about the Union Operator is that you cannot double-count.

(g) [harder] Draw three Venn diagrams illustrating the distributive law $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ one for each of three configurations of A;B; C that you decide. You must draw A;B;C as circles. "Different configurations" means that the circles overlap differently.

Problem 2 Consider the sample space Ω where you flip a fair coin and roll a fair die. (a) [easy] Draw this event space in a box similar to how we did in class.

(b) [easy] What is $|\Omega|$? 12

(c) [easy] Are singleton sets of the outcomes in mutually exclusive? collectively exhaustive?

Singleton sets of the outcomes in Ω are mutually exclusive and collectively exhaustive.

(d) [easy] Does it matter if the coin is flipped before the die, after the die, or simultaneously with the die? Explain in English. It does not matter when the coin is flipped, you will always get a result whether the coin is flipped before, after or simultaneously with the die.

(e) [easy] Consider the set T which represents all outcomes where the coin was flipped tails and E which represents the set of outcomes where the die rolled an even number. Draw a Venn diagram of T and E using circles.

(f) [harder] Describe fully the set 2(E[T)C i.e. list all its elements.

A= $(E \cup T)^c$= {H1, H3, H5}

|A|=3

2^A= {ϕ, {H1},{H3}, {H5}, {H1,H3}, {H3,H5}, {H1,H5}, A}

Problem 3.A "full deck of cards" has 52 cards where each card has two characteristics: (1) one of four suits (2) one of 13 ranks: A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K and each card is unique. The game Euchre

(a) [easy] Let Ω denote the sample space of a Euchre deck and Ω denote the sample space of a full deck. Is $\Omega \subset \Omega$ true? Yes, it is true.

(b) [harder] Construct Ω, the event space of a Euchre deck by using set notation and operations on Ω , the event space of a full deck of cards. Use the "..." notation used in class to specify your sets explicitly and use rank and suit such as 4 to denote the ω's $\in \Omega$. Hint: use the Cartestian product (denoted by x) on two sets.

Ω= {all cards in deck, 52}

Ω= Ω/{2 ...8 ,2 ...8 ,2 ...8 ,2 ...8

(c) [difficult] Let B be the set of black cards, F the set of face cards and the set of spades. Describe the set on the r.h.s of:

(d) [difficult] Do this problem after completing the last questions since it has to do with counting. Given 5 Euchre cards, how many ways is there to order them

$\binom{5}{24}$

(e) [difficult] You are dealt five Euchre cards out of the 24 total hands. How many ways is there to order all hands? 5,100,480 ways

Problem 4 We will review the notation N, Z, Q and R as well as their subsets as introduced in Lecture

(a) [easy] Draw a number line for x and shade in the area that represents the set [1; 3] ∪ (4; 9]. If the set includes a number on the endpoint, draw a solid circle "•" and if does not include the number, draw an open circle "∘"

(b) [easy] Draw a number line for Z ⊂ R where Z := {x ∈ R : |x| 2}. This Z notation we'll be using in a couple months when we get to the normal distribution.

(c) [easy] Draw on the number line the set [0,1] ∩ (0, 1/2) ∩ [0, 1/4].

(d) [harder] Find the set A := [0,1/i]

= [0,1] ∪ [0,0]= {0,1}

(e) [difficult] Find the set B := [0,1/i].

{0}

(f) [easy] Find the set Z\N

{-∞...0}

(g) [harder] Describe the set R\Q as best as you can in English and give an example of an element of this set.

The set of R\Q, or R-Q, would be all of the irrational numbers. I have concluded this because R represents a complete set of all numbers, and if you subtract it by all rational numbers (Q), you will only be left with irrational numbers. An example of an element of this set would be { $\sqrt[2]{}$ }

Problem 5 In this problem, we imagine rolling different sized-dice. Assume the outcomes (each face of each die) are equally likely for that die (see middle of page 9 in H, T and Z for a definition)

Let R be a standard 6-sided die, let S be an 10-sided die, let T be a 12-sided die, and let U be a 18-sided die. What is the sample size of Ω (i.e. $|\Omega|$) for the experiment where we...

(a) [easy] roll R 3 times?

216

(b) [harder] roll R then S then T then U?

12,960

(c) [harder] roll R 34 times, then roll S 45 times, then roll T 12 times, then roll U 76 times.

$6^34 * 10^45 * 12^12 * 18^76$

(d) [harder] Roll R and then roll S only if R rolled greater than or equal to 4. Construct the universe of discourse in this situation by enumerating each outcome of Ω below.

1. {R1} 2. {R2} 3. {R3} 4. {R4,S1} 5. {R4,S2} 6. {R4,S3} 7. {R4,S4} 8. {R4,S5} 9. {R4,S6} 10. {R4,S7} 11. {R4,S8} 12. {R4,S9} 13. {R4,S10} 14. {R5,S1} 15. {R5,S2} 16. {R5,S3} 17. {R5,S4} 18. {R5,S5} 19. {R5,S6} 20. {R5,S7} 21. {R5,S8} 22. {R5,S9} 23. {R5,S10} 24. {R6,S1} 25. {R6,S2} 26. {R6,S3} 27. {R6,S4} 28. {R6,S5} 29. {R6,S6} 30. {R6,S7} 31. {R6,S8} 32. {R6,S9} 33. {R6,S10}

(e) [harder] Would each $\omega \in \Omega$ be equally likely? Is yes, explain why. If no, provide a counterexample.

Yes because if you roll die R and its value is either 4, 5 or 6, there is an equal probability of the outcome when you roll die S after.

Problem 6 Examine the following words and tell me how many permutations there are of the letters. We do not care about keeping track of the individual common letters. For example, in the word dad, there are two d's and we want to treat the permutation d1d2a the same as d2d1a.

(a) [easy] town

24 permutations

(b) [easy] tsktsk (yes, this is a real word!) 6!/3!= 120 Permutations

(c) [harder] mississippi
11!/(4!)(4!)(2!)= 34,650 Permutations

(d) [difficult] supercalifragilisticexpialidocious
34!/(3!)(2!)(2!)(2!)(2!)(3!)(3!)(3!)(7!)(2!)= 1.41246953 * $10^3$0 Permutations

Problem 7 Below is a standard chessboard. Rows one and eight have the following pieces: two rooks, two knights, two bishops, a king and a queen. Rows two and seven have 8 pawns. Rows one and two have all black pieces and rows seven and eight have all white pieces.

(a) [easy] How many ways are there to place the black queen on a white square?
32

(b) [harder] How many ways are there to set up the pieces in the back ranks of both white and black i.e. arrange the two rooks, two knights, two bishops, king and queen on the first row of 8 squares. Note that this game is called "Fischer Random Chess" after the famous grandmaster Bobby Fischer who proposed the idea to make standard chess more exciting.
$(8!/(2!)(2!)(2!))^2$= 25,401,600 ways.

(c) [difficult] The game progresses and white takes two black pawns and black takes two white pawns. How many ways are there to arrange the pieces on the board? We don't care about pieces of a type being unique (i.e. all white pawns are the same, all black rooks are the same, etc)
28!/6!= 4.23456034*$10^2$6

(d) [E.C.] In the answer to (c) are all arrangements "equally likely" during an actual chess game? Explain why or why not.

No, all arrangements are not "equally likely" during an actual chess game because each piece in a game of chess has its own unique way to move on the board.

Problem 8 We have 4 blue marbles, 4 green marbles, 2 orange marbles, and 2 red marbles. For the following questions, if you are using "choose notation", please write your choose notation, then write the formula using factorials, then write the actual number after you compute it.

(a) [easy] Viewing all the marbles as unique, how many ways are there to order the marbles? Note that "order" is another way of saying "permute."

12!= 479,001,600 ways

(b) [harder] Viewing all marbles of the same color as interchangeable, how many ways is there to order the marbles?

12!/(4!)(4!)(2!)(2!)= 207,900 ways

(c) [E.C.] If I pick 4 marbles at random from the collection, how many ways are there to get two-of-a-kind i.e. two marbles of one color and two marbles of a different color.

$\binom{4}{2}\binom{4}{2}+\binom{4}{2}\binom{2}{2}+\binom{4}{2}\binom{2}{2}+\binom{4}{2}\binom{2}{2}+\binom{4}{2}\binom{2}{2}+\binom{2}{2}\binom{2}{2}=$

(4!/2!(4-2)!)(4!/2!(4-2)!)+(4!/2!(4-2)!)(2!/2!(2-2)!)+(4!/2!(4-2)!)(2!/2!(2-2)!)+(4!/2!(4-2)!)(2!/2!(2-2)!)+(4!/2!(4-2)!)(2!/2!(2-2)!)+(2!/2!(2-2)!)(2!/2!(2-2)!)=

36+6+6+6+6+1=

61 ways