Gram-Schmidt Orthogonalization and QR Decomposition

Mohammad Umar Rehman
PhD Student, Control Group,
EE Department, IIT Delhi
umar.ee.iitd@gmail.com

Abstract A very quick and easy to understand introduction to Gram-Schmidt Orthogonalization (Orthonormalization) and how to obtain QR decomposition of a matrix using it. A basic background in linear algebra is assumed.

Aim Given a basis $\mathbb{B}$ of a vector subspace $\mathbf{V}$ spanned by vectors $\{v_1, v_2, \ldots, v_n\}$ it is required to find an orthonormal basis $\{q_1, q_2, \ldots, q_n\}$ (A basis that is orthogonal with unit length of constituent vectors). Thus,

$$q_{\mathrm{i}}^{\mathrm{T}} q_{\mathrm{j}} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Proof. Note that $\mathrm{span}\{v_1, v_2, \ldots, v_n\} = \mathrm{span}\{q_1, q_2, \ldots, q_n\}$ (successive spanning)
Lets start with one vector v_1, normalize this to obtain vector q_1 as

$$q_1 = \frac{v_1}{\|v_1\|} \tag{1}$$

Now, lets include another vector, v_2 to the picture (See Fig. 1) [1]. We have already obtained q_1, now we need q_2, which can be obtained by normalizing a vector that is orthogonal to $v_1(q_1)$. Suppose, this vector is $\tilde{v}_2$. By triangle law of vectors, it can be seen that

$$\tilde{v}_2 = v_2 - v_1$$

But, to be more formal in lines with the algorithm we write $\tilde{v}_2$ as

$$\tilde{v}_2 = v_2 - \mathrm{proj}_{v_1}(v_2) = v_2 - \langle v_2, q_1 \rangle q_1 \tag{2}$$

and,

$$q_2 = \frac{\tilde{v}_2}{\|\tilde{v}_2\|}$$

where, $\langle a, b \rangle = a^{\mathrm{T}} b$ is the inner product.

[1]This is an improved version of the document with the figures included, it would have been better if a successive series of figures were included showing the process step by step. An animation of the GS process can be found on Wikipedia here: `https://en.wikipedia.org/wiki/Gram-Schmidt_process`
Please contact for any clarifications, or found errors. Thanks!

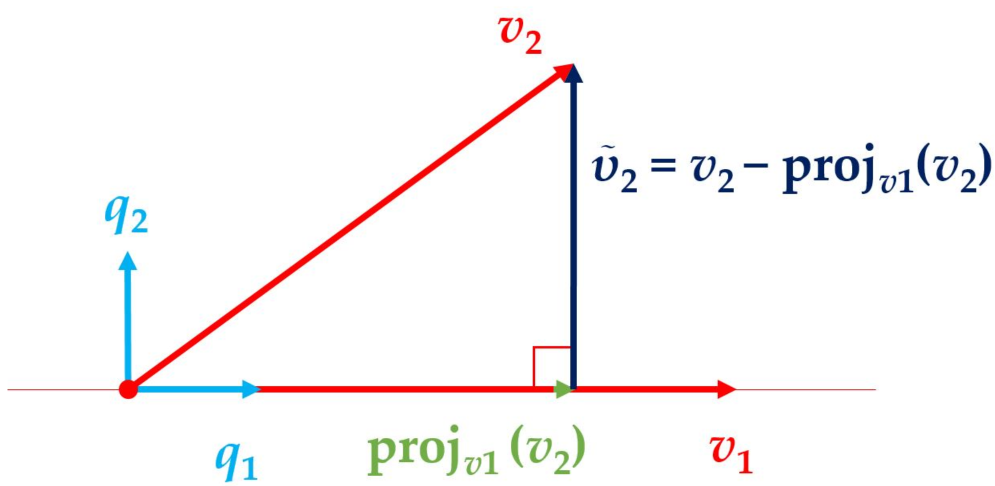

Figure 1: Obtaining an orthonormal basis–2D case

Going one step further we include v_3 and proceed on similar lines to obtain $\tilde{v}_3$ as

$$\begin{aligned} \tilde{v}_3 &= v_3 - \text{proj}_{q_1}(v_3) - \text{proj}_{q_2}(v_3) \\ &= v_3 - \langle v_3, q_1 \rangle q_1 - \langle v_3, q_2 \rangle q_2 \\ q_3 &= \tilde{v}_3 / \|\tilde{v}_3\| \end{aligned} \tag{3}$$

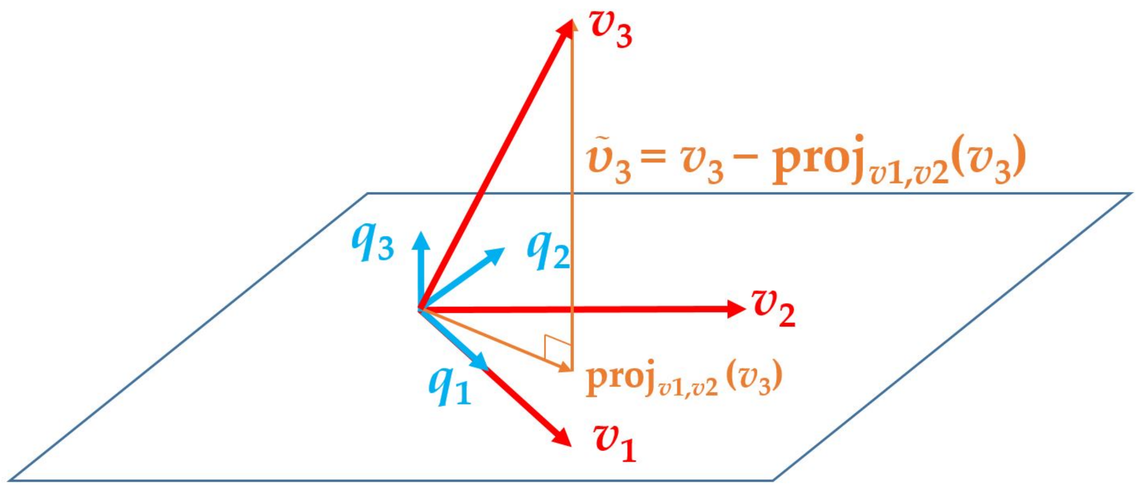

Figure 2: Obtaining an orthonormal basis–3D case

Similarly,

$$\tilde{v}_4 = v_4 - \langle v_4, q_1 \rangle q_1 - \langle v_4, q_2 \rangle q_2 - \langle v_4, q_3 \rangle q_3, \;\; q_4 = \tilde{v}_4 / \|\tilde{v}_4\| \tag{4}$$

So, this process continues and we are in a position to write the expression for $\tilde{v}_n$

$$\tilde{v}_n = v_n - \langle v_n, q_1 \rangle q_1 - \langle v_n, q_2 \rangle q_2 - \cdots - \langle v_n, q_{n-1} \rangle q_{n-1}, \;\; q_n = \tilde{v}_n / \|\tilde{v}_n\| \tag{5}$$

In compact form,

$$\tilde{v}_n = v_n - \sum_{i=0}^{n-1} \langle v_n, q_{\mathrm{i}} \rangle \, q_{\mathrm{i}}$$

□

Hence, we have obtained an orthonormal basis from a regular basis for the vector subspace **V**.

Obtaining QR decomposition
Now, let us rearrange the equations (1) to (5) in terms of v's only

$$\begin{aligned}
v_1 &= \quad \|v_1\| \, q_1 && (6)\\
v_2 &= \langle v_2, q_1 \rangle \, q_1 + \tilde{v}_2 = \langle v_2, q_1 \rangle \, q_1 + \|\tilde{v}_2\| \, q_2 && (7)\\
v_3 &= \langle v_3, q_1 \rangle \, q_1 + \langle v_3, q_2 \rangle \, q_2 + \tilde{v}_3 = \langle v_3, q_1 \rangle \, q_1 + \langle v_3, q_2 \rangle \, q_2 + \|\tilde{v}_3\| \, q_3 && (8)\\
v_4 &= \langle v_4, q_1 \rangle \, q_1 + \langle v_4, q_2 \rangle \, q_2 + \langle v_4, q_3 \rangle \, q_3 + \|\tilde{v}_4\| \, q_4 && (9)
\end{aligned}$$

$$\vdots$$

$$v_n = \langle v_n, q_1 \rangle \, q_1 + \langle v_n, q_2 \rangle \, q_2 + \cdots + \langle v_{n-1}, q_{n-1} \rangle \, q_{n-1} + \|\tilde{v}_n\| \, q_n \tag{10}$$

And rewrite these equations in the matrix form, we get

$$\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} = \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \begin{bmatrix} \|v_1\| & \langle v_2, q_1 \rangle & \langle v_3, q_1 \rangle & \dots & \langle v_n, q_1 \rangle \\ 0 & \|\tilde{v}_2\| & \langle v_3, q_2 \rangle & \dots & \langle v_n, q_2 \rangle \\ 0 & 0 & \|\tilde{v}_3\| & \dots & \langle v_n, q_3 \rangle \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \|\tilde{v}_{n-1}\| & \langle v_{n-1}, q_{n-1} \rangle \\ 0 & 0 & 0 & 0 & \|\tilde{v}_n\| \end{bmatrix} \tag{11}$$

$$\text{or, } V = QR$$

Thus, we have obtained the QR decomposition of the matrix A starting from the Gram-Schmidt process, where Q is an orthonormal matrix and R is an upper triangular matrix.