CPSC 542F WINTER 2017

Convex Analysis and Optimization

Lecture Notes

SUMMER RESEARCH INTERNSHIP, UNIVERSITY OF WESTERN ONTARIO

GITHUB.COM/LAURETHTEX/CLUSTERING

This research was done under the supervision of Dr. Pauline Barmby with the financial support of the MITACS Globalink Research Internship Award within a total of 12 weeks, from June 16th to September 5th of 2014.

First release, August 2014

Contents

1. Convex Sets

1.1 Convexity

1.1.1 Cone

Definition 1.1.1 — Cone. A set $K \in \mathbb{R}^n$, when $x \in K$ implies $\alpha x \in K$.

A non convex cone can be hyper-plane.
For convex cone $x + y \in K, \forall x, y \in K$.
Cone don't need to be "pointed". e.g.
Direct sums of cones $C_1 + C_2 = \{x = x_1 + x_2 | x_1 \in C_1, x_2 \in C_2\}$.

■ **Example 1.1** $S_1^n\{X | X = X^n, \lambda(x) \geq 0\}$
A matrix with positive eigenvalues. ■

Operations preserving convexity

Intersection $C \cap_{i \in \mathbb{I}} C_i$
Linear map Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. If $C \in \mathbb{R}^n$ is convex, so is $A(C) = \{Ax \forall x \in C\}$
Inverse image $A^{-1}(D) = \{x \in \mathbb{R} | Ax \in D\}$

Operations that induce convexity

Convex hull on $S = \cap\{C | S \in C, Cisconvex\}$

■ **Example 1.2** $Co\{x_1, x_2, \cdots, x_m\} = \{\sum_{i=1}^m \alpha_i x_i | \alpha \in \delta_m\}$ ■

For a convex set $x \in C \Rightarrow x = \sum \alpha_i x_i$.

Theorem 1.1.1 — Carathéodory's theorem. If a point $x \in \mathbb{R}^d$ lies in the convex hull of a set P, there is a subset P' of P consisting of $d + 1$ or fewer points such that x lies in the convex hull of P'. Equivalently, x lies in an r-simplex with vertices in P.

1.2 Convex Functions

Definition 1.2.1 — Convex function. Let $C \in \mathbb{R}^n$ be convex, $f : C \to \mathbb{R}$ is convex on f if $x, y \in C \times C$. $\forall \alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y)$

Definition 1.2.2 — Strictly Convex function. Let $C \in \mathbb{R}^n$ be convex, $f : C \to \mathbb{R}$ is strictly convex on f if $x, y \in C \times C$. $\forall \alpha \in (0,1)$, $f(\alpha x + (1-\alpha)y) \langle f(\alpha x) + f((1-\alpha)y)$

Definition 1.2.3 — Strongly convex. $f : C \to \mathbb{R}$ is strongly convex with modules $u \geq 0$ if $f - \frac{1}{2}u||\cdot||^2$ is convex.

Interpretation: There is a convex quadratic $\frac{1}{2}u||\cdot||^2$ that lower bounds f.

■ **Example 1.3** $\min_{x \in C} f(x) \leftrightarrow \min \bar{f}(x)$ Useful to turn this into an unconstrained problem.

$$\bar{f}(x) = \begin{cases} f(x) & if x \in C \\ \infty & elsewhere \end{cases}$$

■

Definition 1.2.4 A function $f : \mathbb{R}^n \to \mathbb{R} \cup \infty \; \bar{\mathbb{R}}$ is convex if $x, y \in \mathbb{R}^n \times \mathbb{R}^n$, $\forall x, y, \bar{f}(\alpha x + (1-\alpha)y) \leq f(\alpha x) + f((1-\alpha)y)$

Definition 1 is equivalent to definition 2 if $f(x) = \infty$.

■ **Example 1.4** $f(x) = \sup_{j \in J} f_j(x)$ ■

1.2.1 Epigraph

Definition 1.2.5 — Epigraph. For $f : \mathbb{R}^n \to \bar{R}$, its epigraph $epi(f) \in \mathbb{R}^{n+1} is the set epi(f) \{(x,\alpha) | f(x) \in \alpha\}$

Next: a function is convex i.f.f. its epigraph is convex.

Definition 1.2.6 A function $f : C \to \mathbb{R}, C \in \mathbb{R}^n$ is convex if $\forall x, y \in C$, $f(ax + (1-a)x) \leq af(x) + (1-a)f(x) \quad \forall a \in (0,1)$.
Strict convex: $x \neq y \Rightarrow f(ax + (1-a)x) \leq af(x) + (1-a)f(x)$

R f is convex $\Rightarrow -f$ is concave.

Level set: $S_\alpha f = \{x | f(x) \leq \alpha\}$.
$S_\alpha f$ is convex $\Leftrightarrow f$ is convex.

Definition 1.2.7 — Strongly convex. $f : C \to \mathbb{R}$ is strongly convex with modules μ if $\forall x, y \in C$, $\forall \alpha \in (0,1)$, $f(ax + (1-a)x) \leq af(x) + (1-a)f(x) - \frac{1}{2\mu}\alpha(1-\alpha)\|x-y\|^2$.

R

- f is 2nd-differentiable, f ix convex $\iff \nabla^2 f(x) \rangle 0$.
- f is strongly convex $\iff \nabla^2 f(x) \rangle \mu I \iff x \geq \mu$

Definition 1.2.8 — 2. $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is convexif $x,y \in \mathbb{R}, \alpha \in (0,1), f(ax+(1-a)x) \leq af(x) + (1-a)f(x)$.

The effective domain of f is $domf = \{x|f(x)\langle+\infty\}$

■ **Example 1.5 — Iudcator function.** $\delta_c(x) = \begin{cases} 0 & x \in C \\ +\infty & elsewhere \end{cases}$.

$dom\delta_c(x) = C$ ■

Definition 1.2.9 — Epigraph. The epigraph of f is $epif = \{(x,\alpha)|f(x) \leq \alpha\}$

The graph of $epif$ is $\{(x,f(x)|x \in domf\}$.

Definition 1.2.10 — III. A function $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is

Theorem 1.2.1 $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ is convex $\iff$ $\forall x,y \in \mathbb{R}^n, \alpha \in (0,1), f(ax+(1-a)x) \leq af(x) + (1-a)f(x)$.

Proof. $\Rightarrow$ take $x,y \in domf$, $(x,f(x)) \in epif$,$(y,f(y)) \in epif$. ■

■ **Example 1.6 — Distance.** Distance to a convexset $d_c(x) = \inf\{\|z-x\||z \in C\}$. Take any two sequence $\{y_k\}and\{\bar{y}_k\} \subset C$ s.t. $\|y_k - x\| \to d_c(x)$, $\|\bar{y}_k - \bar{x}\| \to d_c(\bar{x})$. $z_k = \alpha y_k + (1-\alpha)\bar{y}_k$.

$$\begin{aligned} d_c(\alpha x + (1-\alpha)\bar{x}) &\leq \|z_k - \alpha x - (1-\alpha)\bar{x}\| \\ &= \|\alpha(y_k - x) + (1-\alpha)(\bar{y}_k - \bar{x})\| \\ &\leq \alpha\|y_k - x\| + (1-\alpha)\|\bar{y}_k - \bar{x}\| \end{aligned}$$

Take $k \to \infty$, $d_c(\alpha x + (1-\alpha)\bar{x}) \leq \alpha d(x) + (1-\alpha)d(\bar{x})$ ■

■ **Example 1.7 — Eigenvalues.** Let $X \in S^n := \{n \times nsymmetricmatrix\}$. $\lambda_1(x) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(x)$.
$f_k(x) = \sum_1^n \lambda_i(x)$.
Equivalent characterization

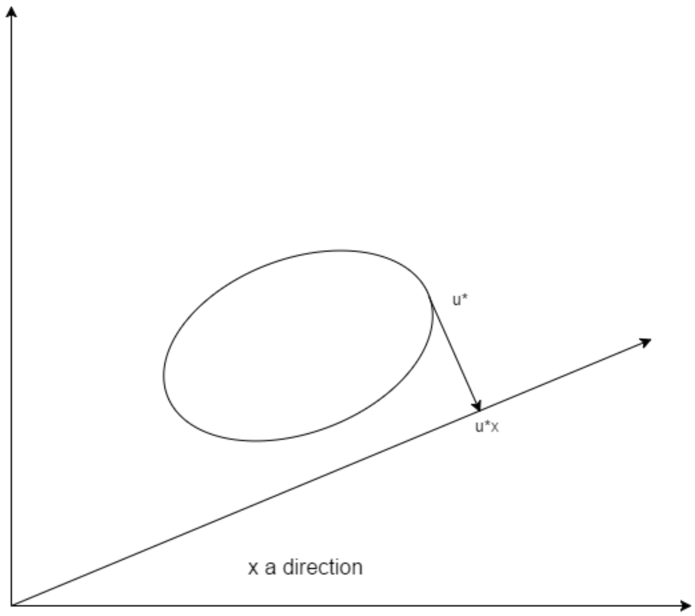
$$\begin{aligned} f_k(x) &= \max\{\sum_i v_i^T X v_i | v_i \perp v_j, i \neq j\} \\ &= \max\{tr(V^T XV|V^T V = I_k\} \end{aligned}$$

$\max\{tr(VV^T X)\}$by circularity

Note $\langle A,B\rangle = tr(A,B)$ is true for symmetric matrix.
$\langle A,A\rangle = |A|_F^2 = \sum_i A_{ii}^2$ ■

1.3 Support Function

Take a set $C \in \mathbb{R}^n$, not necessarily convex.The support function is $\sigma_C = \mathbb{R}^n \to \bar{\mathbb{R}}$. $\sigma_C(x) =$



$\sum\{\langle x,u\rangle | u \in C\}$.

Fact 1.3.1 The support function binds the supporting hyper-plane.

Supporting functions are

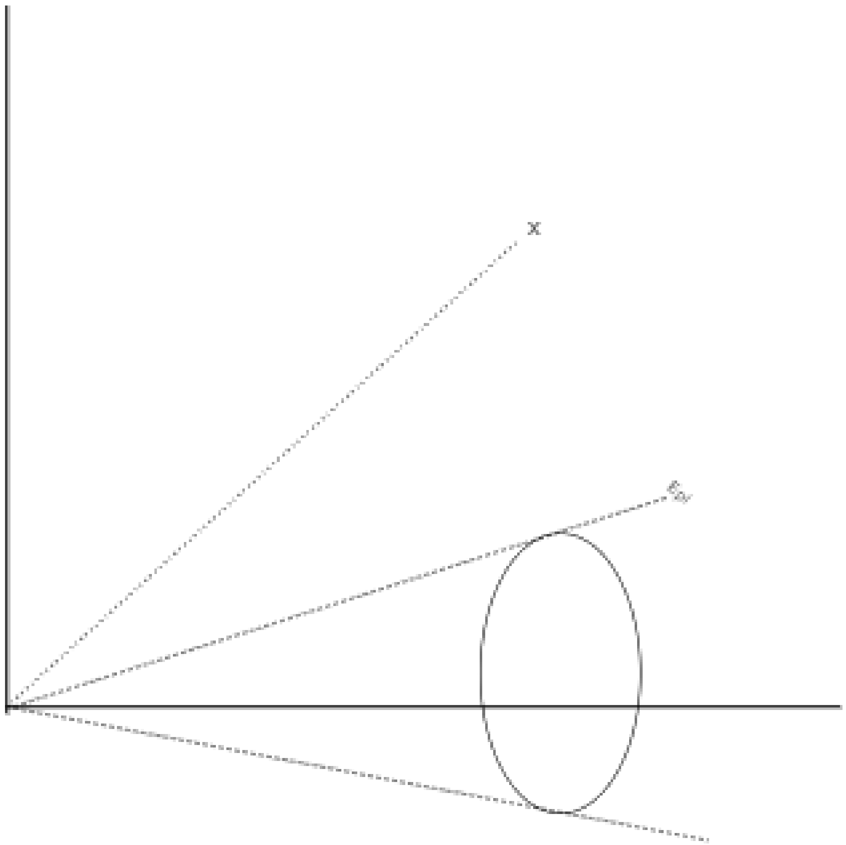
- Positively homogeneous
  $\sigma_C(\alpha x) = \alpha\sigma_C(x)\forall\alpha\rangle 0$
  $\sigma_C(\alpha x) = \sup_{u\in C}\langle\alpha x,u\rangle = \alpha\sup_{u\in C}\langle x,u\rangle = \alpha\sigma_C(x)$
- Sub-linear(a special case of convex, linear combination holds $\forall\alpha$.
  $\sigma_C(\alpha x+(1-\alpha)y) = \sup_{u\in C}\langle\alpha x+(1-\alpha)y,u\rangle \leq \alpha\sup_{u\in C}\langle x,u\rangle + (1-\alpha)\sup_{u\in C}\langle y,u\rangle$

■ **Example 1.8 — L2-norm.** $\|x\| = \sup_{u\in C}\{\langle x,u\rangle, u\in\mathbb{R}^n\}$.
$\|x\|_p = \sup\{\langle x,u\rangle, u\in B_q\}$ where $\frac{1}{p}+\frac{1}{q} = 1$. $B_q = \{\|x\|_q \leq 1\}$.
The norm is

- Positive homogeneous
- sub-linear
- If $0\in C$, σ_C is non-negative.
- If C is central-symmetric, $\sigma_C(0) = 0$ and $\sigma_C(x) = \sigma_C(-x)$

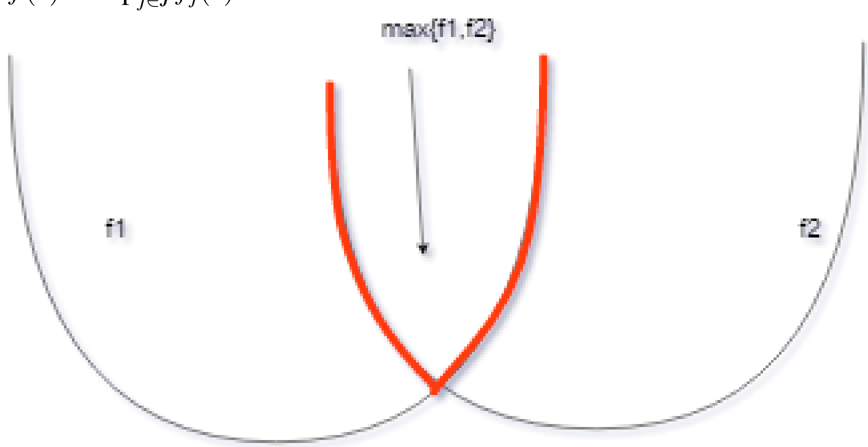
■

Fact 1.3.2 — Epigraph of a support function. $epi\sigma_C = \{(x,t)|\sigma_C(x)\leq t\}$. Suppose $(x,t)\in epi\sigma_C$.
Take any $\alpha > 0$. $\alpha(x,t) = (\alpha x,\alpha t)$.
$\alpha\sigma_C(x) = \alpha\sigma_C(x) \leq \alpha t$. $\alpha(x,c)\in epi\sigma_C$



1.4 Operations Preserve Convexity of Functions

- Positive affine transformation
  $f_1, f_2, \ldots, f_k \in cvx\mathbb{R}^n$.
  $f = \alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_k f_k$
- Supremum of functions. Let $\{f_i\}_{i \in I}$ be arbitrary family of functions. If $\exists x \sup_{j \in J} f_j(x) < \infty \Leftrightarrow$ $f(x) = \sup_{j \in J} f_j(x)$



- Composition with linear map.
  $f \in cvx\mathbb{R}^n, A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. $f \circ A(x) = f(Ax) \in cvx\mathbb{R}^n$

$$
\begin{aligned}
f \circ A(x) &= f(A(\alpha x + (1-\alpha)y)) \\
&= f(A\alpha x + (1-\alpha)Ay) \\
&\leq \alpha f(Ax) + (a-\alpha)f(Ay)
\end{aligned}
$$