ARML: Intermdiate Proofs

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Intermediate Proofs

1.1 Lecture

There are several methods used in Intermediate Proofs:

Contradictions: If we want to show that A is true, we use proof by contradiction by showing that if A is false, then that would result in an impossibility, thereby resulting in A being true.

Induction: Let's say we want to prove a statement $P(n)$ for positive integer n, with n_0 being a fixed positive integer. If $P(n_0)$ is true and $P(k+1)$ is true whenever $P(k)$ is, then $P(n)$ is true for $n \geq n_0$.

Strong Induction: Let's say we want to prove a statement $P(n)$ for positive integers n, with n_0 being a fixed positive integer. If $P(n_0)$ is true and $P(k+1)$ is true whenever $P(m)$ is for $n_0 \leq m \leq k$, then $P(n)$ is true for $n \geq n_0$.

We'll cover these all in depth throughout this lesson.

Example 1.1.1. *Prove that there are infinitely many prime numbers.*

Solution. We proceed by proof by contradiction. Assume that there are only a finite number of prime numbers, namely $p_1, p_2, \cdots, p_k$. Consider the number $M = p_1 p_2 \cdots p_k + 1$. Clearly, M is not divisible by p_i for $1 \leq i \leq k$, therefore M must be divisible by a prime which is not in our assumed set of primes, contradiction. There are therefore infinitely many primes. $\square$

Example 1.1.2. *Prove that there does not exist integers a, b such that $a^2 - 4b = 2$.*

Solution. Assume for the sake of contradiction that there are integers a, b that satisfy the above equation. Rearranging the equation, we see that $a^2 = 2 + 4b = 2(1 + 2b)$. Therefore, a must be even. Let $a = 2a_0$ for some a_0. Substituting this back into the equation gives us

$$(2a_0)^2 = 2(1 + 2b) \implies 4a_0^2 = 2 + 4b \implies 2a_0^2 = 1 + 2b$$

However, $2a_0^2$ and $2b$ are both even, while 1 is not, therefore the above equation is a contradiction mod 2.

Note: Some more experienced problem solvers may have instantly noted that the above equation is a contradiction mod 4 since the possible residues mod 4 are $0, 1$. □

Example 1.1.3. *Prove that $\sqrt{2}$ is irrational.*

Solution. Assume for the sake of contradiction that $\sqrt{2}$ is rational. Therefore $\sqrt{2} = \frac{a}{b}$ for relatively prime a, b. Squaring the equation and multiplying by b^2 on both sides gives us $a^2 = 2b^2$. Therefore, $2 \mid a$ and $a = 2a_0$ for some a_0. Substituting this back into the equation, we have

$$4a_0^2 = 2b^2 \implies 2a_0^2 = b^2$$

Similarly, since the left hand side of the equation is even, b must also be even and $b = 2b_0$ for some b_0. However, $\gcd(a, b) = 2\gcd(a_0, b_0)$, contradicting the assumption that a and b were relatively prime. Contradiction. Therefore $\sqrt{2}$ is irrational. □

Example 1.1.4. *Prove that for $x \in [0, \frac{\pi}{2}]$, $\sin(x) + \cos(x) \geq 1$.*

Solution. Assume for the sake of contradiction that $\sin(x) + \cos(x) < 1$. Squaring this gives

$$(\sin(x) + \cos(x))^2 < 1 \implies \sin^2(x) + \cos^2(x) + 2\sin(x)\cos(x) < 1 \implies 2\sin(x)\cos(x) < 0$$

With the last step following from the Pythagorean Identity that $\sin^2(x) + \cos^2(x) = 1$. However, $x \in [0, \frac{\pi}{2}]$, therefore $2\sin(x)\cos(x) \geq 0$, contradiction. Therefore for $x \in [0, \frac{\pi}{2}], \sin(x) + \cos(x) \geq 1$. □

Example 1.1.5. *Prove the identity* $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$.

Solution. **Base Case:** When $n = 1$, we get $1 + 2^1 = 2^2 - 1$, which is true.

Inductive Hypothesis: Assume that the problem statement holds for $n = k$. We show that it then also holds for $n = k + 1$. Notice that

$$1 + 2 + 2^2 + \cdots + 2^{k+1} = \left(1 + 2 + 2^2 + \cdots + 2^k\right) + 2^{k+1}$$

Now, using the inductive hypothesis, $1+2+\cdots+2^k = 2^{k+1}-1$. Substituting this into the above equation gives us

$$1 + 2 + 2^2 + \cdots + 2^{k+1} = \left(2^{k+1} - 1\right) + 2^{k+1} = 2^{k+2} - 1$$

Our induction is complete, and $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all non-negative n. □

Example 1.1.6. *Prove that* $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$

Solution. **Base Case:** When $n = 1$, $1 \cdot 1! = (1 + 1)! - 1$, which is true.

Inductive Hypothesis: Assume that the problem statement holds for $n = k$. We show that it holds for $n = k + 1$. Notice that

$$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+k\cdot k!+(k+1)\cdot(k+1)! = (1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + k \cdot k!)+(k+1)\cdot(k+1)!$$

Using the inductive hypothesis, $1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k + 1)! - 1$. Substituting this into the above equation,

$$1\cdot 1!+2\cdot 2!+3\cdot 3!+\cdots+(k+1)\cdot(k+1)! = (k+1)!-1+(k+1)\cdot(k+1)! = (k+2)(k+1)!-1 = (k+2)!-1$$

□

Example 1.1.7. *Show that if* n *is a positive integer greater than* 2*, then*

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{3}{5}$$

Solution. Notice that the problem statement says for n being a positive integer **greater than 2**, therefore the base case is 3 rather than 1 (*in the formal definition of induction given above,* $n_0 = 3$).

Base Case: When $n = 3$,

$$\frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60} > \frac{36}{60} = \frac{3}{5}$$

Inductive Hypothesis: Assume the statement holds for $n = k$. Then, we show that it also holds for $n = k + 1$.

Notice that

$$\frac{1}{k+2}+\frac{1}{k+3}+\cdots+\frac{1}{2k+2} = \frac{1}{k+1}+\frac{1}{k+2}+\cdots+\frac{1}{2k}+\left(\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}\right)$$

Using the Inductive Hypothesis, $\frac{1}{k+1} + \frac{1}{k+2} + \cdots + \frac{1}{2k} > \frac{3}{5}$, therefore, substituting this into the above equation gives us

$$\begin{aligned}
\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k+2} &> \frac{3}{5} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1} \\
&= \frac{3}{5} + \frac{1}{2k+1} - \frac{2}{2k+2} + \frac{1}{2k+2} \\
&= \frac{3}{5} + \frac{1}{2k+1} - \frac{1}{2k+2} \\
&= \frac{3}{5} + \frac{1}{(2k+1)(2k+2)}
\end{aligned}$$

Now, using the fact that $\frac{1}{(2k+1)(2k+1)} > 0$, we get

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k+2} > \frac{3}{5} + \frac{1}{(2k+1)(2k+2)} > \frac{3}{5}$$

We are done by induction. □

Example 1.1.8. *The Fibonacci sequence is defined by* $F_1 = F_2 = 1$, *and* $F_n = F_{n-1} + F_{n-2}$ *for all* $n \geq 3$. *Prove that every positive integer* N *can be represented by*

$$N = F_{a_1} + F_{a_2} + \cdots + F_{a_m}$$

for some integers $a_1, a_2, \cdots, a_m$ *satisfying* $2 \leq a_1 < a_2 < \cdots < a_m$.

Solution. The base case of $N = 1 = F_2$ is trivial. To get a feel for the problem, consider the number $N = 79$. How would we go about representing this as a sum of Fibonacci numbers? Well, the smallest Fibonacci number less than 79 is 55. Subtract gives $79 - 55 = 24$. We then repeat this procedure. The smallest Fibonacci number less than 24 is 21. Subtracting yields $24 - 21 = 3$. Finally, $3 = 2 + 1 = F_3 + F_2$. Therefore, $79 = 55 + 21 + 3 + 1 = F_{10} + F_8 + F_3 + F_2$.

We think of how to generalize this method. In a regular induction problem, we would assume that it holds for $N = K$ and show that it holds for $N = K + 1$. However, in the above example, once we subtract 55 we are left with a number close to K but less than it. This therefore queues for us to use strong induction.

Inductive Hypothesis: Assume that the problem statement holds for all positive integers from 1 to K. We show that the problem statement holds for $K + 1$.

Let F_a be the largest Fibonacci number with $F_a \leq K + 1$. If $F_a = K + 1$, then we are clearly done. Otherwise, $F_a < K + 1 < F_{a+1}$, therefore

$$0 < (K + 1) - F_a < F_{a+1} - F_a = Fa - 1$$

Now, by our inductive hypothesis, $(K + 1) - F_a = F_{b_1} + F_{b_2} + \cdots + F_{b_m}$. Furthermore, since $(K + 1) - F_a < F_{a-1}$, we have that $2 \leq b_1 < b_2 < \cdots < b_m < a$. Therefore, $K + 1 = F_a + F_{b_1} + F_{b_2} + \cdots + F_{b_m}$ satisfies the desired condition. □

1.2 Problems for the Reader

Problem 1.2.1. Prove that $\sqrt[3]{3}$ is irrational.

Problem 1.2.2. Prove that there are infinitely many primes of the form $4k + 3$.

Problem 1.2.3. Prove that if $a^2 - 2a + 7$ is even, then a must be odd.

Problem 1.2.4. Prove that the product of 5 consecutive integers is divisible by 120.

Problem 1.2.5. Prove that the number $\log_2 3$ is irrational.

Problem 1.2.6. Prove that if $4 \mid (a^2 + b^2)$ and a and b are both positive integers, then a and b cannot both be odd.

Problem 1.2.7. Prove that there are no rational roots to the equation $x^3 + x + 1 = 0$.

Problem 1.2.8. Prove that there are no $(x, y) \in \mathbb{Q}^2$ (*meaning x and y are rational*) such that $x^2 + y^2 - 3 = 0$.

Problem 1.2.9. Prove that if a, b, c are odd integers, then the equation $ax^2 + bx + c = 0$ does not have any integer roots.

Problem 1.2.10. Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Problem 1.2.11. Prove that

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \frac{(m+2)!}{2!} + \cdots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)}$$

Problem 1.2.12. The kth triangular number is equivalent to $\frac{k(k+1)}{2}$. Prove that the sum of the first n triangular numbers is $\frac{n(n+1)(n+2)}{6}$.

Problem 1.2.13. Show that if n is a positive integer, then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$.

Problem 1.2.14. Use induction and/or telescoping sums to prove that $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.

Problem 1.2.15. The sequence $x_1, x_2, x_3, \cdots$ is defined by $x_1 = 2$ and $x_{k+1} = x_k^2 - x_k + 1$ for all $k \geq 1$. Find $\sum_{k=1}^{\infty} \frac{1}{x_k}$.

Problem 1.2.16. Prove that $n^4 \leq 4^n$ for all positive integers n greater than 3.

Problem 1.2.17. Let $x + \frac{1}{x} = a$, for some integer a. Prove that $x^n + \frac{1}{x^n}$ is an integer for all $n \geq 0$.

Problem 1.2.18. Show that the nth Fibonacci number, $F_n = \binom{n-1}{0} + \binom{n-1}{1} + \cdots$

Problem 1.2.19. On a large, flat field n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd show that there is at least one person left dry. Is this always true when n is even?

Problem 1.2.20. Prove that for all natural n, that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$.